49. Green's Theorem

Let $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ be a vector field in R^2, and suppose C is a path that starts and ends at the same point such that it does not cross itself. Such a path is called a *simple closed loop*, and it will enclose a region R. Assume M and N and its first partial derivatives are defined within R including its boundary C. Furthermore, the path is to be traversed (circulated) in a counterclockwise direction, called the *positive orientation*. If these conditions are met, then the line integral around the simple loop path may be evaluated by a double integral. This is called **Green's Theorem**, and is written

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y)\ dA.$$

If $\mathbf{F}$ is a conservative vector field, then $M_y = N_x$, so that the integrand $N_x - M_y = 0$. Thus, in a conservative vector field, all line integrals along a simple closed loop path evaluate to 0. In a physical sense, there is no net circulation around the loop, and a conservative vector field is often called a *rotation-free* (or *irrotational*) vector field.

When calculating a line integral, you should check two things:

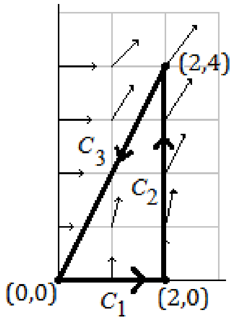
- Is the vector field conservative?
- Is the path a simple closed loop?

The following table will help plan the calculation accordingly.

	$\mathbf{F}$ is conservative	$\mathbf{F}$ is not conservative
C is a simple closed loop	0	Use Green's Theorem
C is not a loop of any kind (it has different start and end points).	Find the potential function $\varphi(x, y)$ and calculate the line integral by the Fundamental Theorem of Line Integrals (The FTLI)	Parameterize the path(s) in variable t and calculate the line integral directly.

Example 49.1: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=\langle y,4x\rangle$ and C is a triangle, traversed from (0,0) to (2,0) to (2,4) back to (0,0).

Solution: Sketch C and observe that it is a simple closed loop that is traversed counterclockwise:



To evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ as a sequence of line integrals, we would need to divide the path into three smaller paths: C_1 being the line from (0,0) to (2,0), C_2 being the line from (2,0) to (2,4), and C_3 being the line from (2,4) to (0,0).

For C_1, we have $\mathbf{r}_1(t)=\langle 2t,0\rangle$ with $0\le t\le 1$, so that $\mathbf{r}_1'(t)=\langle 2,0\rangle$ and $\mathbf{F}(t)=\langle 0,8t\rangle$. Thus, $\mathbf{F}\cdot d\mathbf{r}_1=\langle 0,8t\rangle\cdot\langle 2,0\rangle=0$, so that $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1=0$. In the above image, note that the vector field elements are orthogonal to the segment C_1.

For C_2, we have $\mathbf{r}_2(t)=\langle 2,4t\rangle$ with $0\le t\le 1$, so that $\mathbf{r}_2'(t)=\langle 0,4\rangle$ and $\mathbf{F}(t)=\langle 4t,8\rangle$. Thus, $\mathbf{F}\cdot d\mathbf{r}_2=\langle 4t,8\rangle\cdot\langle 0,4\rangle=32$, so that $\int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2=\int_0^1 32\,dt=[32t]_0^1=32$. The vector field elements agree with the direction of C_2.

For C_3, we have $\mathbf{r}_3(t)=\langle 2-2t,4-4t\rangle$ with $0\le t\le 1$, so that $\mathbf{r}_3'(t)=\langle -2,-4\rangle$ and $\mathbf{F}(t)=\langle 4-4t,8-8t\rangle$. Thus, $\mathbf{F}\cdot d\mathbf{r}_3=\langle 4-4t,8-8t\rangle\cdot\langle -2,-4\rangle=40t-40$, which gives $\int_{C_3}\mathbf{F}\cdot d\mathbf{r}_3=\int_0^1(40t-40)\,dt=[20t^2-40t]_0^1=-20$. The vector field elements disagree (point against) the direction of C_3.

Since

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2+\int_{C_3}\mathbf{F}\cdot d\mathbf{r}_3\,,$$

we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 32 - 20 = 12.$$

Using Green's Theorem, we find that $N_x - M_y = 4 - 1 = 3$, so that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R 3 \; dA \\ &= 3 \iint_R dA \\ &= 3(4) \\ &= 12.\end{aligned}$$

The constant integrand was moved to the front, leaving $\iint_R dA$, which is the area of region *R*. Using geometry, the area of *R*, a triangle with base 2 and height 4, is $\frac{1}{2}(2)(4) = 4$.

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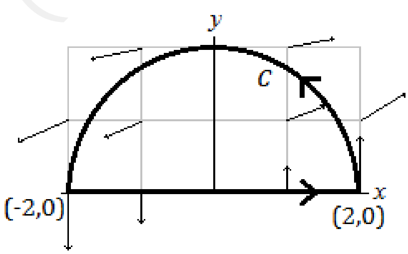
Example 49.2: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2xy, x\rangle$ and *C* traverses from (2,0) to (–2,0) along a semi-circle of radius 2, centered at the origin, in the counter-clockwise direction, then from (–2,0) back to (2,0) along a straight line.

Solution: Path *C* is a simple closed loop traversed in a counterclockwise direction, as shown below.



To find $\int_C \mathbf{F} \cdot d\mathbf{r}$, we use Green's Theorem. Since the region R is a semicircle of radius 2, we will evaluate the double integral using polar coordinates.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (N_x - M_y)\ dA \\ &= \iint_R (1 - 2x)\ dA \\ &= \int_0^{\pi} \int_0^2 (1 - 2r\cos\theta)\, r\, dr\, d\theta \\ &= \int_0^{\pi} \int_0^2 (r - 2r^2\cos\theta)\, dr\, d\theta.\end{aligned}$$

The inside integral is evaluated with respect to r:

$$\int_0^2 (r - 2r^2\cos\theta)\, dr = \left[\frac{1}{2}r^2 - \frac{2}{3}r^3\cos\theta\right]_0^2 = 2 - \frac{16}{3}\cos\theta.$$

This is then integrated with respect to θ:

$$\int_0^{\pi} \left(2 - \frac{16}{3}\cos\theta\right)\, d\theta = \left[2\theta - \frac{16}{3}\sin\theta\right]_0^{\pi} = 2\pi.$$

Thus, the line integral along C is $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$. There is positive circulation along this path induced by the vector field.

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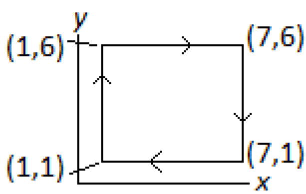
Example 49.3: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 3y, -x + y\rangle$ and C traverses a rectangle from (1,1) to (1,6) to (7,6) to (7,1) back to (1,1).

Solution: A sketch of the path C shows it to be a simple closed loop traversed in a *clockwise* direction. In order to use Green's Theorem, we would traverse it in the counterclockwise direction, which is equivalent to traversing each segment in its opposite direction. This means that we will multiply our result by –1 to account for this "opposite" direction.



Using Green's Theorem, we have $N_x - M_y = -1 + (-3) = -4$:

$$\iint_R (N_x - M_y)\ dA = \iint_R (-4)\ dA = -4\int_1^7 \int_1^6 dy\, dx.$$

The double integral $\int_1^7 \int_1^6 dy\, dx$ is the area of the rectangle, which is (6)(5) = 30. Thus,

$$\iint_R (N_x - M_y)\ dA = -4(30) = -120.$$

However, since *C* was traversed in the opposite direction, we negate this result. We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 120.$$

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Example 49.4: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 5x^4 + y^2, 2yx \rangle$ and *C* is an ellipse with major axis of 12 along the *x*-axis, and minor axis of 8 along the *y*-axis, in a counter-clockwise direction.

Solution: Using Green's Theorem, we have

$$\iint_R (N_x - M_y)\ dA = \iint_R (2y - 2y)\ dA = \iint_R 0\ dA = 0.$$

Note that **F** is conservative, since $M_y = N_x$. There is no need to parameterize the ellipse.

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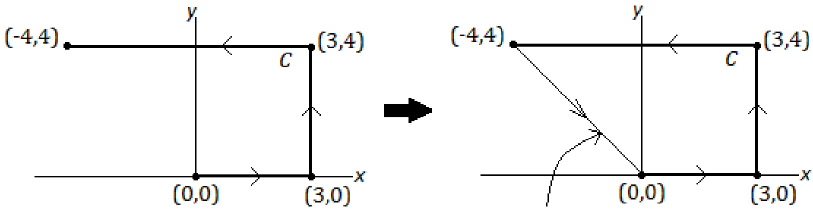
Green's Theorem can be used to find the line integral of a non-loop path. We "close off" the path forming a loop, as this next example shows:

Example 49.5: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2y, x^2 \rangle$ and *C* is a sequence of line segments from (0,0) to (3,0) to (3,4) to (–4,4).

Solution: The path *C* is not a simple closed loop. Thus, we would have to parametrize each line segment one at a time and determine the value of each line integral individually. Instead, we can add in the final line segment, that from (–4,4) to (0,0), thus creating a simple closed loop traversed counter-clockwise.



The path C is not a closed loop.

Add in the final segment to make a closed loop.

Using Green's Theorem, we have

$$\iint_R \left(N_x - M_y\right)\ dA = \int_0^4 \int_{-y}^3 (2x-2)\ dx\ dy = 2\int_0^4 \int_{-y}^3 (x-1)\ dx\ dy.$$

The inside integral is first evaluated:

$$\begin{aligned}\int_{-y}^3 (x-1)\ dx &= \left[\frac{1}{2}x^2 - x\right]_{-y}^3 \\ &= \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2}y^2 + y\right) \\ &= -\frac{1}{2}y^2 - y + \frac{3}{2}.\end{aligned}$$

This result is then integrated with respect to y:

$$2\int_0^4 \left(-\frac{1}{2}y^2 - y + \frac{3}{2}\right)\ dy = \left[-\frac{1}{3}y^3 - y^2 + 3y\right]_0^4 = -\frac{76}{3}.$$

We now need to evaluate the line integral from (–4,4) to (0,0), the segment that we added in to form the closed loop.

We have $\mathbf{r}(t) = \langle -4+4t, 4-4t\rangle$, where $0 \le t \le 1$. Thus, $\mathbf{r}'(t) = \langle 4, -4\rangle$ and $\mathbf{F}(t) = \langle 2(4-4t), (-4+4t)^2\rangle = \langle 8-8t, 16t^2 - 32t + 16\rangle$. Along this path segment, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle 8-8t, 16t^2-32t+16\rangle \cdot \langle 4,-4\rangle\ dt \\ &= \int_0^1 (-64t^2 + 96t - 32)\ dt \\ &= \left[-\frac{64}{3}t^3 + 48t^2 - 32t\right]_0^1 \\ &= -\frac{16}{3}.\end{aligned}$$

Therefore, the line integral from (0,0) to (3,0) to (3,4) to (–4,4) is the value we found from Green's Theorem, $-\frac{76}{3}$, subtracted by the value of the line integral along the segment we used to "close off" the region, $-\frac{16}{3}$. We then have $\int_C \mathbf{F}\cdot d\mathbf{r} = -\frac{76}{3} - \left(-\frac{16}{3}\right) = -\frac{60}{3} = -20$.

As a check, here are the individual line integrals along the three line segments:

From (0,0) to (3,0): We have $\mathbf{r}(t) = \langle 3t, 0\rangle$ with $0 \le t \le 1$. Thus, $\mathbf{r}'(t) = \langle 3,0\rangle$ and $\mathbf{F}(t) = \langle 0,9t^2\rangle$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 0,9t^2\rangle \cdot \langle 3,0\rangle\, dt = \int_0^1 0\, dt = 0$.

From (3,0) to (3,4): We have $\mathbf{r}(t) = \langle 3,4t\rangle$, with $0 \le t \le 1$. Thus, $\mathbf{r}'(t) = \langle 0,4\rangle$ and $\mathbf{F}(t) = \langle 8t, 9\rangle$, and therefore $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 8t, 9\rangle \cdot \langle 0,4\rangle\, dt = \int_0^1 36\, dt = [36t]_0^1 = 36$.

From (3,4) to (–4,4): We have $\mathbf{r}(t) = \langle 3-7t, 4\rangle$, with $0 \le t \le 1$. Thus, $\mathbf{r}'(t) = \langle -7,0\rangle$ and $\mathbf{F}(t) = \langle 8, (3-7t)^2\rangle$, and therefore, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 8, (3-7t)^2\rangle \cdot \langle -7,0\rangle\, dt = \int_0^1 -56\, dt = [-56t]_0^1 = -56$.

The sum is $0 + 36 - 56 = -20$, which agrees with our earlier answer.

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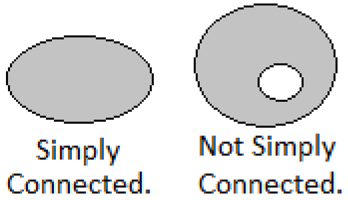
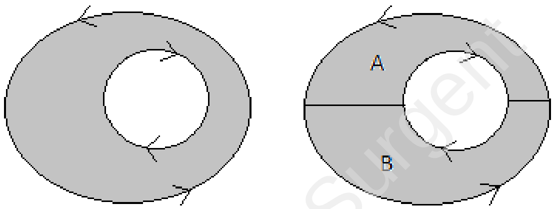
If C is a simple closed loop, then the region R bounded by C is **simply connected**. All of the regions in the preceding examples in this section are simply connected. A simply connected region in the plane has no holes.



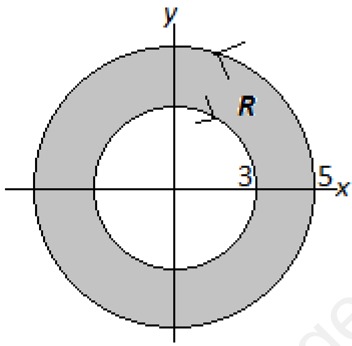
Green's Theorem requires a simply connected region R. However, a non-simply connected region can be made into two (or more) simply connected regions by dividing the region carefully.



In the above image, a non-simply connected region is strategically divided into two subregions, A and B, that are each simply connected. Notice that the counterclockwise circulation is preserved in both cases. The line integrals along the two "cuts" will cancel, since the flow is in opposite directions depending on whether A or B is being considered. Green's Theorem can then be applied to each subregion, and often combined into one double integral covering the entire region.

Example 49.6: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle e^x + 2y, 3x - \sin y\rangle$ and C is the boundary of a region R enclosed by two concentric circles, centered at the origin, one of radius 5 and the other of radius 3. Assume the circulation in the outer circle is counterclockwise, and that the circulation on the inner circle is clockwise.

Solution: The region R and its boundary C are shown below.



Using Green's Theorem, we have $N_x - M_y = 3 - 2 = 1$. Using polar coordinates, the region R can be defined as $3 \le r \le 5$ and $0 \le \theta \le 2\pi$. Thus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\int_3^5 1\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_3^5 d\theta \\
&= 8\int_0^{2\pi} d\theta \qquad \left(\left[\frac{r^2}{2}\right]_3^5 = 8\right) \\
&= 8(2\pi) = 16\pi.
\end{aligned}$$

Note that $\int_0^{2\pi}\int_3^5 1\,r\,dr\,d\theta$ is the area of R represented as a double integral, so we can verify using geometry. The area inside a circle of radius 5 is 25π, and the area inside a circle of radius 3 is 9π, and their difference is 16π.