Outline of the Proof of the Divergence Theorem

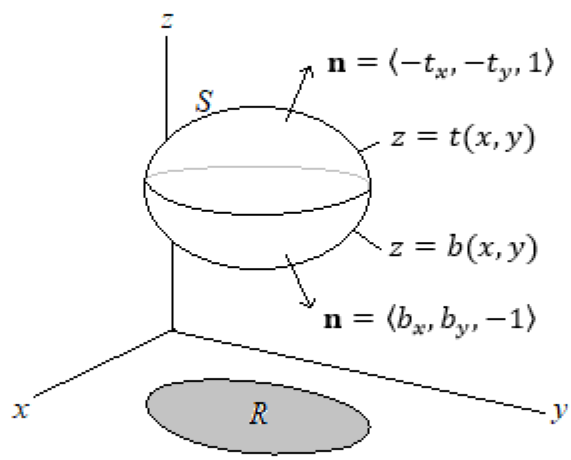
Let $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ re a vector field in R^3, and let S be a surface that encloses a subregion *V* of R^3. Assume that this subregion is of finite volume, has no "tendrils" extending to infinity, has no interior hollows or voids, is connected (no "islands"), and is topologically well-behaved (has a distinct inside and outside, no odd behaviors such as the Klein 4-bottle (look it up)).

In such a case, the flux of **F** through *S* can be calculated by a triple integral over *V*:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{F}\, dV.$$

Consider a simple subregion of R^3 as shown to the right. Relative to the *xy*-plane, this subregion is split in half at its equator, forming a top surface, $z = t(x, y)$, and a bottom surface, $z = b(x, y)$, each defined over a common region *R* in the *xy*-plane.

Note that the normal vector **n** must point away from the interior. Thus, for the top surface, we have $\mathbf{n} = \langle -t_x, -t_y, 1\rangle$, and for the bottom surface, we have $\mathbf{n} = \langle b_x, b_y, -1\rangle$.



Writing $\mathbf{F}(x, y, z)$ in **i**-**j**-**k** form, we have $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The dot product with **n** gives:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}) \cdot \mathbf{n}\, dS$$
$$= \iint_S (M(x, y, z)\mathbf{i} \cdot \mathbf{n} + N(x, y, z)\mathbf{j} \cdot \mathbf{n} + P(x, y, z)\mathbf{k} \cdot \mathbf{n})\ dS.$$

Look at each term one at a time. Start with the last one, $\iint_S P(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS$. First, split it into its top and bottom surfaces, where *S* is the union of *t* and *b*:

$$\iint_{S=t\cup b} P(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_t P(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS + \iint_b P(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS.$$

Now, replace the *z*'s in each integral and the normal vectors **n** with the forms involving *t* and *b*:

$$\iint_t P(x, y, t(x, y))\mathbf{k} \cdot \langle -t_x, -t_y, 1\rangle\, dy\, dx + \iint_b P(x, y, b(x, y))\mathbf{k} \cdot \langle b_x, b_y, -1\rangle\, dy\, dx.$$

Since these are defined over a common region R in the xy-plane, we can write this as one integral again:

$$\iint_R \left[P(x,y,t(x,y))\mathbf{k}\cdot\langle -t_x, -t_y, 1\rangle + P(x,y,b(x,y))\mathbf{k}\cdot\langle b_x, b_y, -1\rangle\right] dy\, dx.$$

Now, perform the dot product. Note that only the **k** component is being "dotted" with the two **n** vectors. This gives

$$\iint_R \left[P\big(x,y,t(x,y)\big) - P\big(x,y,b(x,y)\big)\right] dy\, dx.$$

The expression $P\big(x,y,t(x,y)\big) - P\big(x,y,b(x,y)\big)$ can be written as the result of an integral in which $t(x,y)$ was the top bound, $b(x,y)$ was the bottom bound, and the function $P(x,y,z)$ was the result of integrating P_z. Thus, "working backwards", we have

$$P\big(x,y,t(x,y)\big) - P\big(x,y,b(x,y)\big) = \int_{b(x,y)}^{t(x,y)} \frac{\partial P}{\partial z}\, dz.$$

This means that

$$\iint_R \left[P\big(x,y,t(x,y)\big) - P\big(x,y,b(x,y)\big)\right] dy\, dx = \iint_R \left[\int_{b(x,y)}^{t(x,y)} \frac{\partial P}{\partial z}\, dz\right] dy\, dx.$$

Simplified, the last integral is

$$\iiint_V P_z\, dV.$$

This shows that

$$\iint_S P(x,y,z)\mathbf{k}\cdot\mathbf{n}\, dS = \iiint_V P_z\, dV.$$

This process is repeated twice more by splitting the region relative to the *xz* and *yz* planes. This shows that

$$\iint_S M(x,y,z)\mathbf{i}\cdot\mathbf{n}\, dS = \iiint_V M_x\, dV \qquad \textit{and} \qquad \iint_S N(x,y,z)\mathbf{j}\cdot\mathbf{n}\, dS = \iiint_V N_y\, dV.$$

Summing, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \left(M_x + N_y + P_z\right) dV = \iiint_V \text{div}\,\mathbf{F}\, dV.$$