Calculus-III Directional Derivatives Practice Problems.

Answers are not included. You are encouraged to work together and post ideas and comments on Piazza.

Example: Find the slope of the tangent line to $f(x,y) = x^3 - xy^2$ at $x_0 = 2$ and $y_0 = 5$ in the direction of $\mathbf{v} = \langle 4,7 \rangle$.

Solution: The direction *must* be a unit vector:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \langle \frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}} \rangle.$$

Thus, the components are $u_1 = \frac{4}{\sqrt{65}}$ and $u_2 = \frac{7}{\sqrt{65}}$.

Now, differentiate and evaluate at the given *x* and *y* values:

$$\begin{matrix} f_x(x,y) = 3x^2 - y^2 \\ f_y(x,y) = -2xy \end{matrix} \rightarrow \begin{matrix} f_x(2,5) = 3(2)^2 - (5)^2 = -13 \\ f_y(2,5) = -2(2)(5) = -20 \end{matrix}$$

The general form of the directional derivative is

$$D_u f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$$

Assemble your ingredients:

$$D_u f(2,5) = -13\left(\frac{4}{\sqrt{65}}\right) + (-20)\left(\frac{7}{\sqrt{65}}\right) = -\frac{192}{\sqrt{65}} \approx -23.814 \ldots.$$

This is a very steep negative (downward) slope.

Be careful! The direction can be expressed multiple ways. Please read each problem carefully.

1. Find the slope of the tangent line to $f(x,y) = 2x + y^4$ at $x_0 = 1$ and $y_0 = 3$ in the direction of $\mathbf{v} = \langle 2,1 \rangle$.
2. Find the slope of the tangent line to $f(x,y) = x^3 y^2$ at $x_0 = -1$ and $y_0 = 4$ in the direction of $\mathbf{v} = \langle 5,12 \rangle$.
3. Find the slope of the tangent line to $f(x,y) = (x + 2y^2)^2$ at $x_0 = 2$ and $y_0 = -3$ in the direction of $\mathbf{v} = \langle 7,24 \rangle$.
4. Find the slope of the tangent line to $f(x,y) = \sqrt{x^2 + y^4}$ at $x_0 = 3$ and $y_0 = 1$ in the direction of the point (5,7).
5. Find the slope of the tangent line to $f(x,y) = 6 - x^2 + xy - y^2$ at $x_0 = 4$ and $y_0 = 2$ in the direction of the point (7,1).
6. Find the slope of the tangent line to $f(x,y) = xe^y$ at $x_0 = 4$ and $y_0 = \ln 5$ in the direction of the origin.
7. Find the slope of the tangent line to $f(x,y) = \frac{x+6y^2}{y^3}$ at $x_0 = 1$ and $y_0 = 3$ in the direction that is opposite the origin.
8. Find the slope of the tangent line to $f(x,y) = 2x + 3y + y^4$ at $x_0 = 0$ and $y_0 = 2$ in the direction of a ray that has an angle of $\frac{\pi}{3}$ radians relative to the positive *x*-direction.

9. Find the slope of the tangent line to $f(x,y) = 5x + 2y - 10$ at $x_0 = 2$ and $y_0 = 5$ in the northeast direction (using the usual orientation from a map)
10. Find the slope of the tangent line to $f(x,y) = x^2y^3z^4$ at $x_0 = 2$, $y_0 = 1$ and $z_0 = -3$ in the direction of $\mathbf{v} = \langle 5,1,3 \rangle$.

Gradients

The gradient of $f(x,y)$ is $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$. It is a vector-valued function.

On any smooth and continuous (differentiable) surface, the gradient at x_0 and y_0 always points in the direction of maximum increase. This direction need not be a unit vector.

The magnitude of the gradient, $|\nabla f(x_0, y_0)|$, is the slope of the maximum increase. It is a scalar.

The direction of maximum decrease and its slope arte just the negations of the gradient and its magnitude.

Example: Find the direction and the slope of maximum ascent for $f(x,y) = x^3 - xy^2$ at $x_0 = 2$ and $y_0 = 5$.

Solution: The gradient is

$$\nabla f(x,y) = \langle 3x^2 - y^2, -2xy \rangle.$$

Evaluated at the given coordinates, we have

$$\nabla f(2,5) = \langle -13, -20 \rangle.$$

This is the direction of maximum increase. The slope is the magnitude:

$$m = |\langle -13, -20 \rangle| = \sqrt{(-13)^2 + (-20)^2} = \sqrt{569} \approx 23.8537 \ldots$$

Problems: Find the direction and the slope of maximum increase (ascent) for the functions and the points given in #1-10 of the previous set of problems on this sheet.