52. The Del Operator: Divergence and Curl

Let $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ be a vector field in R^3. The **del operator** is represented by the symbol ∇, and is written

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \qquad \text{or} \qquad \nabla = \langle \partial_x, \partial_y, \partial_z \rangle.$$

By itself, the del operator is meaningless. It must be combined with a vector field **F** via a dot product or cross product to be meaningful. For example, the del operator can be combined with a vector field **F** as a dot product:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \mathbf{F}(x, y, z) \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle \\ &= \frac{\partial}{\partial x} M(x, y, z) + \frac{\partial}{\partial y} N(x, y, z) + \frac{\partial}{\partial z} P(x, y, z) \end{aligned}$$

This is called the **divergence** of **F**, and is written shorthand as div $\mathbf{F} = \nabla \cdot \mathbf{F} = M_x + N_y + P_z$. In R^2, we have div $\mathbf{F} = \nabla \cdot \mathbf{F} = M_x + N_y$.

div **F** is a scalar function.

The del operator can also be combined with **F** as a cross product:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \left(P_y - N_z\right)\mathbf{i} - (P_x - M_z)\mathbf{j} + \left(N_x - M_y\right)\mathbf{k}.$$

This is called the **curl** of **F**, and is written shorthand as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle.$$

curl **F** is a vector field.

The second component, $M_z - P_x$, is simplified slightly by distributing the leading negative. Also note that the third component, $N_x - M_y$, is the integrand for Green's Theorem. Thus, we will see that Green's Theorem is a special case of a higher-dimension analog called Stokes' Theorem that uses curl **F**.

Example 52.1: Given $\mathbf{F}(x, y, z) = \langle xy^2, x^2yz^2, 2xz^4 \rangle$, find div $\mathbf{F}$ and curl $\mathbf{F}$.

Solution: For div $\mathbf{F}$, we have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= M_x + N_y + P_z \\ &= y^2 + x^2z^2 + 8xz^3. \end{aligned}$$

For curl $\mathbf{F}$, we have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 0 - 2x^2yz, 0 - 2z^4, 2xyz^2 - y^2 \rangle \\ &= \langle -2x^2yz, -2z^4, 2xyz^2 - y^2 \rangle. \end{aligned}$$

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Example 52.2: Given $\mathbf{F}(x, y, z) = \langle 2y, xz, x + 2y \rangle$, find div $\mathbf{F}$ and curl $\mathbf{F}$.

Solution: For div $\mathbf{F}$, we have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= M_x + N_y + P_z \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

For curl $\mathbf{F}$, we have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 2 - x, -1, z - 2 \rangle. \end{aligned}$$

When div $\mathbf{F} = 0$, the vector field is **incompressible**.

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Be careful with the syntax when using the symbol ∇. If f is a scalar function, then ∇f is the gradient of f. If $\mathbf{F}$ is a vector field, then $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$, and $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$. However, statements like $\nabla\mathbf{F}$ and $\nabla \cdot f$ have no meaning. On the other hand, statements like $\nabla \cdot \nabla f$, $\nabla \times \nabla f$ and $\nabla \cdot (\nabla \times \mathbf{F})$ are well-defined.

Example 52.3: Given $\mathbf{F}(x, y, z) = \langle 2xyz^3,\ x^2z^3,\ 3x^2yz^2 \rangle$, find div **F** and curl **F**.

Solution: For div **F**, we have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= M_x + N_y + P_z \\ &= 2yz^3 + 6x^2yz. \qquad \text{(Note that } N_y = 0\text{)} \end{aligned}$$

For curl **F**, we have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 3x^2z^2 - 3x^2z^2, 6xyz^2 - 6xyz^2, 2xz^3 - 2xz^3 \rangle \\ &= \langle 0,0,0 \rangle = \mathbf{0}. \end{aligned}$$

When curl **F** = **0**, the vector field is **irrotational**.

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Example 52.4: Given $\mathbf{F}(x, y, z) = \langle 2,1,-4 \rangle$, find div **F** and curl **F**.

Solution: For div **F**, we have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= M_x + N_y + P_z \\ &= 0. \end{aligned}$$

For curl **F**, we have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 0 - 0, 0 - 0, 0 - 0 \rangle \\ &= \langle 0,0,0 \rangle = \mathbf{0}. \end{aligned}$$

A constant vector field is both incompressible (div **F** = 0)
and irrotational (curl **F** = **0**).

The divergence operator is used to show (quantify) how a vector field flows through a region bounded by permeable membranes. The region can then be made as small as we desire, down to a point. Thus, divergence can show the existences of a **source** (where, roughly speaking, the flow radiates away from the point), or a **sink** (where a flow collects into a point).

If the divergence of a vector field **F** is 0, then there are no sources nor sinks in **F**. If a certain amount of mass flows into a region, then the same amount must flow away from the region in order to maintain the balance, and thus, the flow is incompressible. The flow of fluid, as modeled by a vector field **F**, is a good example of an incompressible field. It is not possible to compress an idealized fluid. On the other hand, heat or gasses can be compressed, allowing for sources and/or sinks.

The curl operator is used to show (quantify) the tendency for the vector field **F** to create "spin", and this spin is defined around a vector representing the axis of spin, at any given point. Thus, in a vector field **F**, there is super-imposed another vector field, curl **F**, which consists of vectors that serve as axes of rotation for any possible "spinning" within **F**. In a physical sense, "spin" creates circulation, and curl **F** is often used to show how a vector field might induce a current through a wire or loop immersed within that field. If curl **F** = **0**, then the vector field **F** induces no spin (or circulation).

Curl can be defined on a vector field within R^2, as shown below:

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Example 52.5: Given $\mathbf{F}(x, y) = \langle xy, 2x^2\rangle$. Find curl **F**.

Solution: Rewrite **F** to include a third component of 0:

$$\mathbf{F}(x, y, z) = \langle xy, 2x^2, 0\rangle.$$

Thus,

$$\begin{aligned}\text{curl }\mathbf{F} &= \nabla \times \mathbf{F}\\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle\\ &= \langle 0,0, N_x - M_y\rangle\\ &= \langle 0,0,3x\rangle.\end{aligned}$$

A vector field **F** confined to the *xy*-plane (R^2) may induce a spin, and if so, all axes of rotations point into the third dimension, orthogonal to the *xy*-plane. At each point within some bounded region in R^2, there may be a spin. While some spins may cancel others, the net result will be evident at the boundary, where such spins then induce a current around that boundary.

Curl **F** may also be used to show if **F** is conservative. In general, if curl **F** = **0**, then **F** is (usually) conservative. We then find a possible potential function $\phi(x,y,z)$ such that $\nabla\phi = \mathbf{F}$.

Example 52.6: Given $\mathbf{F}(x,y,z) = \langle 2xyz^3, x^2z^3, 3x^2yz^2\rangle$, show that curl **F** = **0**, and find a potential function $\phi(x,y,z)$ such that $\nabla\phi = \mathbf{F}$.

Solution: From an earlier example, we showed that curl **F** = **0**:

$$\begin{aligned}\text{curl }\mathbf{F} &= \nabla \times \mathbf{F}\\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle\\ &= \langle 3x^2z^2 - 3x^2z^2, 6xyz^2 - 6xyz^2, 2xz^3 - 2xz^3\rangle\\ &= \langle 0,0,0\rangle = \mathbf{0}.\end{aligned}$$

This suggests that **F** is probably conservative. We seek a potential function by antidifferentiating *M* with respect to *x*, *N* with respect to *y*, and *P* with respect to *z*, and examining the results:

$$\int 2xyz^3\,dx = x^2yz^3; \quad \int x^2z^3\,dy = x^2yz^3; \quad \int 3x^2yz^2\,dz = x^2yz^3.$$

Observe that all three antiderivatives result in x^2yz^3. We check by showing that $\nabla x^2yz^3 = \mathbf{F}$. It is, and thus, $\phi(x,y,z) = x^2yz^3$ is a potential function of **F**, so that **F** is a conservative vector field in R^3. In this case, **F** is also called a **gradient vector field**.

In general, if a function $f(x,y,z)$ has continuous second-order derivatives over the relevant domain, then ∇f is a gradient vector field, and curl $\nabla f = \nabla \times \nabla f = \mathbf{0}$.

Furthermore, if given $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ and assuming *M*, *N* and *P* have continuous first-ordered partial derivatives, then div curl **F** = ∇ · (∇ × **F**) = 0.

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Example 52.7: Given $\mathbf{F}(x,y,z) = \langle xy^2, x^2yz^2, 2xz^4\rangle$, verify that div curl **F** = 0.

Solution: From an earlier example, curl $\mathbf{F} = \langle -2x^2yz, -2z^4, 2xyz^2 - y^2\rangle$. Thus,

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{F}) &= \frac{\partial}{\partial x}(-2x^2yz) + \frac{\partial}{\partial y}(-2z^4) + \frac{\partial}{\partial z}(2xyz^2 - y^2)\\ &= -4xyz + 0 + 4xzy\\ &= 0.\end{aligned}$$

Example 52.8: Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where

$$\mathbf{F}(x,y,z)=\left\langle \frac{2x}{z},\frac{1}{z},-\frac{x^2+y}{z^2}\right\rangle,$$

and C is is a line segment from (2,1,3) to (4,4,4), then another line segment from (4,4,4) to (5,7,6).

Solution: It is possible that **F** is a conservative (gradient) vector field. We find curl **F**:

$$\begin{aligned}\text{curl}\,\mathbf{F}&=\langle P_y-N_z, M_z-P_x, N_x-M_y\rangle\\&=\langle -\frac{1}{z^2}-\left(-\frac{1}{z^2}\right),-\frac{2x}{z^2}-\left(-\frac{2x}{z^2}\right),0-0\rangle\\&=\langle 0,0,0\rangle=\mathbf{0}.\end{aligned}$$

Since curl **F** = **0**, then **F** is likely conservative. We now find $\phi(x,y,z)$ such that $\nabla\phi=\mathbf{F}$. We antidifferentiate M with respect to x, N with respect to y, and P with respect to z, and examine the results:

$$\int\frac{2x}{z}\,dx=\frac{x^2}{z};\quad \int\frac{1}{z}\,dy=\frac{y}{z};\quad \int\left(-\frac{x^2+y}{z^2}\right)dz=\frac{x^2+y}{z}.$$

The union of these terms is

$$\phi(x,y,z)=\frac{x^2+y}{z}.$$

This is a potential function since $\phi_x=M=\frac{2x}{z}$, $\phi_y=N=\frac{1}{z}$, and $\phi_z=P=-\frac{x^2+y}{z^2}$.

Thus, the line integral can be determined by using the Fundamental Theorem of Line Integrals, and avoiding the need to parameterize the line segments:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r}&=\left[\frac{x^2+y}{z}\right]_{(2,1,3)}^{(5,7,6)}\\&=\left(\frac{(5)^2+(7)}{(6)}\right)-\left(\frac{(2)^2+(1)}{(3)}\right)\\&=\frac{32}{6}-\frac{5}{3}=\frac{11}{3}.\end{aligned}$$

Visualizing Divergence

To see divergence at a point, visualize a small box around that point, then infer whether more mass is entering into this box than leaving, more is leaving the box than entering, or equal amounts are flowing into and out of the box. For example, suppose a point P, shown below, is located within a vector field represented by arrows:

A box is drawn around P, and we see that the vectors "entering" from the left are the same magnitude as those "leaving" to the right. If we scale the box down and assume a similar behavior of the vector field for these smaller boxes, then it is reasonable to infer that in this case, there are equal amounts of material entering as leaving. Thus, there is no divergence at P.

In the image below, the arrows differ in magnitude, but it is still evident that there are equal amounts of material entering as leaving. There is no divergence at P.

In the next image, it appears more material is entering than is leaving. Thus, at P, there is negative divergence, and P is a sink.

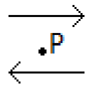
In this image, more material is leaving than is entering, so at P, there is positive divergence and P is a source:

Visualizing Curl

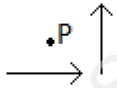
Curl is the tendency of a vector field to cause a spin at a point, the spin rotating around an axis of revolution. However, when viewing a vector field, "seeing" curl is not as obvious. It should not be confused with any apparent "curviness" of a vector field. A fluid may flow along a non-straight line path, yet have no curl.

To see evidence of curl at a point P, look for vectors that seem to shear (face opposite directions) near P, or look for any concentric behavior of the flow lines. However, even this won't strongly indicate curl.

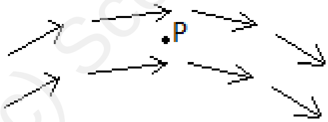
For example, in the image below, there is probably a non-zero curl vector at P. Note the opposing directions of the vector field.



In the next image, there is probably non-zero curl at P as well:

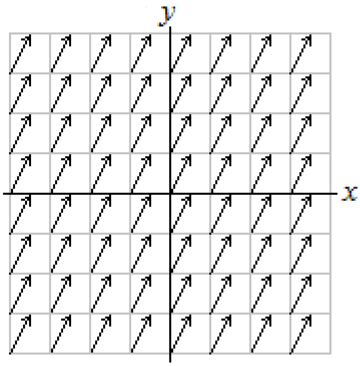


However, in the next image, there is possibly no curl (zero curl) at P:



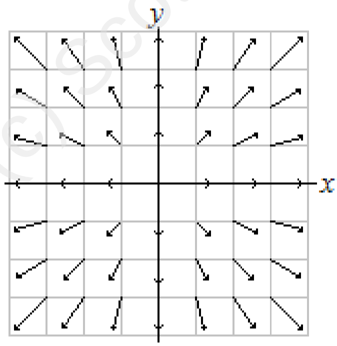
The formula for curl **F** allows us to definitively quantify the curl at any given point, which is helpful since viewing it from an image of a vector field may be difficult.

The following are examples of vector fields and their divergence and curl:



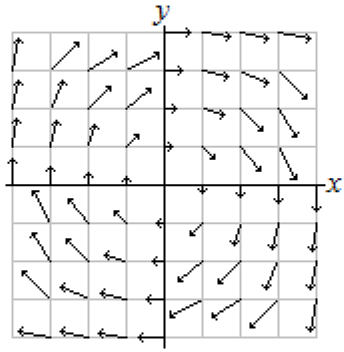
$$\mathbf{F}(x, y) = \langle 1,2 \rangle$$
$$\text{div } \mathbf{F} = 0$$
$$\text{curl } \mathbf{F} = \mathbf{0}.$$

Constant vector fields have no divergence and no curl.



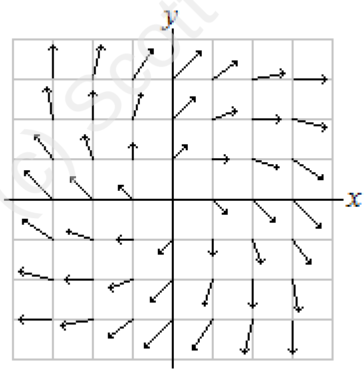
$$\mathbf{F}(x, y) = \langle x, y \rangle$$
$$\text{div } \mathbf{F} = 2$$
$$\text{curl } \mathbf{F} = \mathbf{0}.$$

All vectors emanate away from the origin and grow in magnitude. Draw a small box anywhere and note that more mass is moving "out" than entering "in". All points in the plane are considered "sources". Divergence is positive. There is no rotation, so curl is 0.



$$\mathbf{F}(x, y) = \langle y, -x\rangle$$
$$\text{div } \mathbf{F} = 0$$
$$\text{curl } \mathbf{F} = \langle 0,0,-2\rangle.$$

This field has no divergence, but it does have curl. Since curl is negative, the spin is clockwise, and the curl vectors point in the negative z direction ("into" the page).



$$\mathbf{F}(x, y) = \langle x + y, y - x\rangle$$
$$\text{div } \mathbf{F} = 2$$
$$\text{curl } \mathbf{F} = \langle 0,0,-2\rangle.$$

There is divergence at all points: draw a small box anywhere and note that more mass is moving "out" than entering "in". There is also curl: note the general clockwise "spiral" nature of the vector field. A point anywhere in the plane would be compelled to spin.