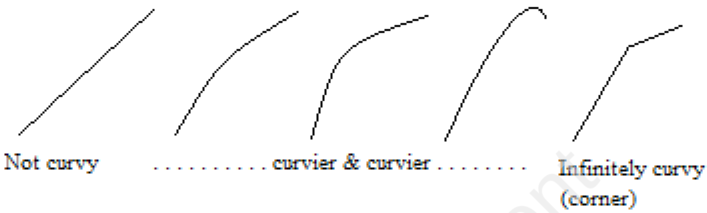
21. Curvature

Let C be a continuous path in R^n. The expression $\left|\frac{d\mathbf{T}}{ds}\right|$ represents the rate of change in a unit tangent vector per unit segment along the path—roughly, how "fast" $\mathbf{T}$ turns per unit segment s. This is called **curvature**, and it offers a way to quantify the "curviness" of a path. Intuitively, we would expect that a line has no (or zero) curvature, while a path with high curvature would turn quickly, in the extreme case appearing as a corner in the sense that the turn happened instantaneously.



Also, it is plausible (and as it turns out, it is correct) to assume that a circle has constant curvature. Small circles have high curvature since the turning is happening faster, while big circles have small curvature. It is then plausible to assume there is an inverse relationship between the radius of a circle and its curvature. That is, for a circle of radius r, its curvature, denoted κ (lower-case letter *kappa*), should be $\frac{1}{r}$. As we will see in an example, this is true: $\kappa = \frac{1}{r}$.

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)|$, we have

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{\left|\frac{d}{dt}\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)\right|}{|\mathbf{r}'(t)|}.$$

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Example 21.1: Find the curvature of a circle with radius r.

Solution: Parameterize the circle: $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$, with $0 \le t \le 2\pi$. Differentiating, we have

$$\mathbf{r}'(t) = \langle -r\sin t, r\cos t\rangle$$

and

$$|\mathbf{r}'(t)| = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = \sqrt{r^2(\cos^2 t + \sin^2 t)} = r.$$

Thus,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -r\sin t\,, r\cos t\rangle}{r} = \langle -\sin t\,, \cos t\rangle.$$

Differentiating $\mathbf{T}(t)$, we have

$$\frac{d}{dt}\mathbf{T}(t) = \frac{d}{dt}\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) = \langle -\cos t\,, -\sin t\rangle.$$

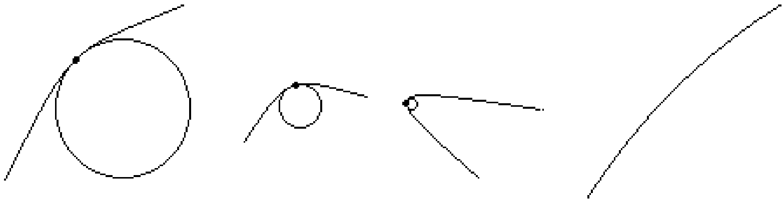
Its magnitude is

$$\left|\frac{d}{dt}\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1.$$

Recalling that $|\mathbf{r}'(t)| = r$, the curvature of a circle of radius r is

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{\left|\frac{d}{dt}\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)\right|}{|\mathbf{r}'(t)|} = \frac{1}{r}.$$

The implications of this result allow us to infer that a large circle has small curvature, so that zero curvature would be appropriate for a line (roughly speaking, a circle with a very large radius would have sides of imperceptible curvature, so allowing the radius to tend to infinity as a limit, the arc of such a circle would be a line in the limiting sense). Similarly, a very small circle has a very large curvature, and taking this to its logical extreme, a circle of zero radius would have infinite curvature, and such a concept would be appropriate for a path with a corner.



Circles of osculation: as the circles decrease in radius (left three images), they "fit" paths of greater curvature. On the right is a portion of a circle with a very large radius. It appears to be nearly linear. Note that curvature can change in value for different points along a path.

Example 21.2: Find the curvature of a line, $y = mx + b$.

Solution: Letting $x = t$, the parameterization of the line is $\mathbf{r}(t) = \langle t, mt + b\rangle$. Thus, $\mathbf{r}'(t) = \langle 1, m\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1+m^2}$. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, m\rangle}{\sqrt{1+m^2}}.$$

Note that this is a constant vector. Therefore, $\frac{d}{dt}\mathbf{T}(t) = 0$ and as a result, the curvature of a line is $\kappa(t) = 0$, as expected.

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Example 21.3: Find the curvature of the parabola $y = x^2$.

Solution: We have $\mathbf{r}(t) = \langle t, t^2\rangle$, so that $\mathbf{r}'(t) = \langle 1,2t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$. Thus,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1,2t\rangle}{\sqrt{1+4t^2}} = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right\rangle.$$

Differentiating, we have

$$\frac{d}{dt}\mathbf{T}(t) = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right\rangle.$$

Its magnitude is

$$\begin{aligned}\left|\frac{d}{dt}\mathbf{T}(t)\right| &= \sqrt{\left(-\frac{4t}{(1+4t^2)^{3/2}}\right)^2 + \left(\frac{2}{(1+4t^2)^{3/2}}\right)^2} \\ &= \sqrt{\frac{16t^2}{(1+4t^2)^3} + \frac{4}{(1+4t^2)^3}} \\ &= \sqrt{\frac{4+16t^2}{(1+4t^2)^3}} \\ &= \sqrt{\frac{4(1+4t^2)}{(1+4t^2)^3}} \\ &= \frac{2}{1+4t^2}.\end{aligned}$$

Thus, the curvature is

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{\left(\frac{2}{1+4t^2}\right)}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

Since a parabola does not have constant curvature, it is no surprise that the curvature can vary as a function of t. At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$. Thus, a circle of osculation at the origin would have a radius of $\frac{1}{2}$. As t trends away from 0, the curvature decreases to 0; equivalently, the circles of osculation become larger. This should be plausible, as the parabola is "curviest" at its vertex, and less "curvy" farther away.

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Curvature should not be confused with concavity.

Finding the curvature of a curve defined parametrically can involve many steps. An equivalent formula for determining curvature of a path in R^2 given by $y = f(t)$ is

$$\kappa(t) = \frac{y''}{(1+(y')^2)^{3/2}},$$

assuming that its first and second derivative exist.

Notation can vary: κ (kappa) is often used in situations where the sign of the curvature is ignored (i.e. always taken as a non-negative value). Since it is possible for $y'' < 0$, then curvature may take on negative values in which case lower-case k is used to represent signed curvature. However, using κ in these situations is usually acceptable.

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Example 21.4: Find the curvature of the parabola $y = t^2$ using the alternative formula given above.

Solution: We have $y' = 2t$ and $y'' = 2$. Thus,

$$\kappa(t) = \frac{y''}{(1+(y')^2)^{3/2}} = \frac{2}{(1+(2t)^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}.$$