47. Conservative Vector Fields

Given a function $z = \phi(x, y)$, its gradient is $\nabla\phi = \langle\phi_x, \phi_y\rangle$. Thus, $\nabla\phi$ is a **gradient** (or **conservative**) **vector field**, and the function ϕ is called a **potential function**.

Suppose we are given the vector field first, in the form $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$. Can we show that this is a conservative vector field? Recall that $\phi_{xy} = \phi_{yx}$ is true by Clairaut's Theorem. Assuming such a function ϕ exists, we infer that $\phi_x = M$ and that $\phi_y = N$, and observing that $\phi_{xy} = \phi_{yx}$, this is equivalent to showing that $M_y = N_x$. In other words, if ϕ exists, then $M_y = N_x$, and if $M_y = N_x$ is true, then ϕ exists. (The exceptions to this property are rare and not relevant to this discussion).

To summarize, if given a vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$, then two cases result:

- If $M_y = N_x$, then **F** is conservative, and there exists a potential function ϕ.
- If $M_y \neq N_x$, then **F** is not conservative and no such potential function ϕ exists.

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Example 47.1: Determine if $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y\rangle$ is conservative. If it is, find a potential function $\phi(x, y)$ such that $\nabla\phi = \mathbf{F}$.

Solution: From **F**, we have $M(x, y) = 3x^2y^2$ and $N(x, y) = 2x^3y$. We find M_y and N_x:

$$M_y = 6x^2y \quad \text{and} \quad N_x = 6x^2y.$$

Since $M_y = N_x$, then **F** is conservative, and there exists a function $\phi(x, y)$ such that $\phi_x = 3x^2y^2$ and $\phi_y = 2x^3y$. Since we assume that $\phi_x = 3x^2y^2$, we integrate it with respect to *x*:

$$\int 3x^2y^2\,dx = x^3y^2 + g(y).$$

Here, $g(y)$ represents a possible term in variable *y*, noting that under differentiation with respect to *x*, $\frac{\partial}{\partial x}g(y) = 0$. Now differentiate this result with respect to *y*:

$$\frac{\partial}{\partial y}\left(x^3y^2 + g(y)\right) = 2x^3y + g'(y).$$

This is compared to $N(x, y) = 2x^3y$:

$$2x^3y + g'(y) = 2x^3y.$$

This suggests $g'(y) = 0$ so that integrating, $g(y) = k$, a constant. Thus, the potential function has the form $\phi(x, y) = x^3y^2 + k$. In such cases, any constants are set to 0. This leaves

$$\phi(x, y) = x^3y^2$$

as a potential function of **F**. This is easily checked by showing that $\phi_x = 3x^2y^2$ and $\phi_y = 2x^3y$.

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Example 47.2: Determine if $\mathbf{F}(x, y) = \langle xy, 1 - x^2 \rangle$ is conservative. If it is, find the potential function $\phi(x, y)$ such that $\nabla\phi = \mathbf{F}$.

Solution: We have $M(x, y) = xy$ and $N(x, y) = 1 - x^2$. Observe that $M_y = x$ and $N_x = -2x$. Since $M_y \neq N_x$, vector field **F** is not conservative, and there does not exist a function whose gradient is **F**.

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Example 47.3: Determine if $\mathbf{F}(x, y) = \langle y - 3, x + 2 \rangle$ is conservative. If it is, find a potential function ϕ.

Solution: We have $M(x, y) = y - 3$ and $N(x, y) = x + 2$. Observe that $M_y = 1$ and $N_x = 1$. Since $M_y = N_x$, the vector field **F** is conservative. To determine $\phi(x, y)$, we first integrate $M(x, y)$ with respect to *x*:

$$\int (y - 3)\, dx = xy - 3x + g(y).$$

Differentiating this result with respect to *y*, we have

$$\frac{\partial}{\partial y}\big(xy - 3x + g(y)\big) = x + g'(y).$$

This is compared to $N(x, y) = x + 2$:

$$x + g'(y) = x + 2.$$

Thus, $g'(y) = 2$, so that $g(y) = 2y + k$ (any constants of integration can be set to 0). A potential function is

$$\phi(x, y) = xy - 3x + 2y,$$

which we check by showing that $\nabla\phi(x, y) = \mathbf{F}(x, y)$:

$$\phi_x(x, y) = y - 3, \qquad \phi_y(x, y) = x + 2.$$

These are precisely the components of **F**, so $\nabla\phi(x, y) = \mathbf{F}(x, y)$.

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> Given a conservative vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$, a "shortcut" to find a potential function $\phi(x, y)$ is to integrate $M(x, y)$ with respect to *x*, and $N(x, y)$ with respect to *y*, and to form the union of the terms in each antiderivative. However, check that the alleged potential function's partial derivatives with respect to *x*, and respect to *y*, do give *M* and *N*, respectively.

Example 47.4: Given the conservative vector field $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y\rangle$, find a potential function, $\phi(x, y)$.

Solution: Integrate $M(x, y) = 3x^2y^2$ with respect to *x*, and $N(x, y) = 2x^3y$ with respect to *y*:

$$\int 3x^2y^2\, dx = x^3y^2 \quad \text{and} \quad \int 2x^3y\, dy = x^3y^2.$$

Observing the two antiderivatives, we infer that $\phi(x, y) = x^3y^2$ may be a potential function. A check that $\phi_x = M(x, y) = 3x^2y^2$ and $\phi_y = N(x, y) = 2x^3y$ shows that this function is a correct potential function. Any constants of integration can be ignored.

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Example 47.5: Given the conservative vector field

$$\mathbf{F}(x, y) = \langle 3x^2 + 2y, 2x - 2y\rangle,$$

find the potential function, $\phi(x, y)$.

Solution: Integrate $3x^2 + 2y$ with respect to *x*, and $2x - 2y$ with respect to *y*:

$$\int (3x^2 + 2y)\, dx = x^3 + 2xy \quad \text{and} \quad \int (2x - 2y)\, dy = 2xy - y^2.$$

The union of terms from these two antiderivatives is $\phi(x,y) = x^3 + 2xy - y^2$. We check that this is actually a potential function: $\phi_x = 3x^2 + 2y = M(x,y)$, and $\phi_y = 2x - 2y = N(x,y)$. Thus, this is a correct potential function.

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Example 47.6: A student is given the vector field $\mathbf{F}(x,y) = \langle x^2, xy \rangle$. He then integrates x^2 with respect to x, getting $\int x^2\,dx = \frac{1}{3}x^3$, and integrates xy with respect to y, getting $\int xy\,dy = \frac{1}{2}xy^2$. He concludes that the potential function is $\phi(x,y) = \frac{1}{3}x^3 + \frac{1}{2}xy^2$. Explain the error.

Solution: Vector field **F** is *not* conservative since $M_y \neq N_x$. Thus, there is no potential function that generates **F**. Note that the alleged potential function, $\phi(x,y) = \frac{1}{3}x^3 + \frac{1}{2}xy^2$, does not generate **F** since $\phi_x = x^2 + \frac{1}{2}y^2$, which is not equal to x^2. In other words, $\nabla\phi \neq \mathbf{F}$.

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Vector fields in R^3 can also be conservative, where $w = \phi(x,y,z)$ is a potential function of a vector field $\nabla\phi = \mathbf{F}(x,y,z) = \langle \phi_x, \phi_y, \phi_z \rangle$. However, showing that a vector field **F** in R^3 is conservative is found by showing that curl **F** = **0**. The **curl** of a vector field is discussed in Section 52.

48. Fundamental Theorem of Line Integrals

If **F** is a conservative vector field in R^2 with $\phi(x,y)$ as its potential function, and *C* is a directed path with endpoints $a = (x_0, y_0)$ and $b = (x_1, y_1)$, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= [\phi(x,y)]_a^b \\ &= \phi(b) - \phi(a) \\ &= \phi(x_1, y_1) - \phi(x_0, y_0).\end{aligned}$$

This is called the **Fundamental Theorem of Line Integrals** (FTLI). In this case, there is no need to parametrize the path, as the value of the line integral depends only on the potential function evaluated at the endpoints, then subtracted in the usual manner of integration.

A couple of corollaries follow:

• Line integrals in a conservative vector field are *path independent*, meaning that any path from *a* to *b* will result in the same value of the line integral.

• If the path *C* is a simple loop, meaning it starts and ends at the same point and does not cross itself, and **F** is a conservative vector field, then the line integral is 0.

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Example 48.1: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle 3x^2y^2, 2x^3y \rangle$ and *C* is the line segment from *a* = (1,2) to *b* = (4, –3).

Solution: From a previous example, we showed that **F** is conservative, and that a potential function is $\phi(x,y) = x^3y^2$. Therefore,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= [x^3y^2]_{(1,2)}^{(4,-3)} \\ &= (4)^3(-3)^2 - (1)^3(2)^2 \\ &= 576 - 4 = 572.\end{aligned}$$

Note that we did not actually parametrize the line segment to solve this line integral.

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Example 48.2: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=\langle 2x,3y\rangle$ and C is any path from a = (1,0) to b = (0,1).

Solution: Let's try a few common paths. Suppose C is a line from a to b. We have $\mathbf{r}(t)=\langle 1-t,t\rangle$, where $0\le t\le 1$. Thus, $\mathbf{r}'(t)=\langle -1,1\rangle$ and $\mathbf{F}(t)=\langle 2(1-t),3t\rangle$. The line integral is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle 2(1-t),3t\rangle\cdot\langle -1,1\rangle\ dt\\ &= \int_0^1 (5t-2)\,dt\\ &= \left[\frac{5}{2}t^2-2t\right]_0^1\\ &= \frac{5}{2}-2\\ &= \frac{1}{2}.\end{aligned}$$

Now suppose C is a quarter circle, centered at the origin, with radius 1. It is parametrized as $\mathbf{r}(t)=\langle \cos t,\sin t\rangle$, where $0\le t\le \frac{\pi}{2}$. As a result, $\mathbf{r}'(t)=\langle -\sin t,\cos t\rangle$ and $\mathbf{F}(t)=\langle 2\cos t,3\sin t\rangle$. The line integral is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/2} \langle 2\cos t,3\sin t\rangle\cdot\langle -\sin t,\cos t\rangle\,dt\\ &= \int_0^{\pi/2} \sin t\cos t\,dt\\ &= \left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2} \qquad \begin{cases}\text{Letting } u=\sin t\\ \text{so that } du=\cos t\ dt.\end{cases}\\ &= \frac{1}{2}.\end{aligned}$$

Suppose C is a parabola $x=1-y^2$. It is parametrized as $\mathbf{r}(t)=\langle 1-t^2,t\rangle$, where $0\le t\le 1$. Thus, $\mathbf{r}'(t)=\langle -2t,1\rangle$ and $\mathbf{F}(t)=\langle 2(1-t^2),3t\rangle$. The line integral is

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 \langle 2(1-t^2),3t\rangle\cdot\langle -2t,1\rangle\,dt$$

$$= \int_0^1 (4t^3 - t)\, dt$$

$$= \left[t^4 - \frac{1}{2}t^2\right]_0^1$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}.$$

It appears that regardless the path, the line integral is $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}$. Although three examples are not a "proof" that this assertion is true, it suggests that it might be worth approaching the problem from a different perspective.

Observe that the vector field **F** is conservative: $M(x, y) = 2x$, so that $M_y = 0$, and $N(x, y) = 3y$, so that $N_x = 0$. A potential function is $\phi(x, y) = x^2 + \frac{3}{2}y^2$ (You should verify this). Thus, using the Fundamental Theorem of Line Integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[x^2 + \frac{3}{2}y^2\right]_{(1,0)}^{(0,1)}$$

$$= \left((0)^2 + \frac{3}{2}(1)^2\right) - \left((1)^2 + \frac{3}{2}(0)^2\right)$$

$$= \frac{3}{2} - 1$$

$$= \frac{1}{2}.$$

This example illustrates that in a conservative vector field, the line integral along any path between two fixed endpoints will always give the same result. Rather than try many different paths, it's easier to first check whether **F** is conservative. If it is, then skip the parametrization step entirely, and proceed to finding a potential function and using the Fundamental Theorem of Line Integrals.

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Example 48.3: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y, x + 2y\rangle$ and C is a sequence of line segments from (1,3) to (2,7) to (–4,0) to (8,2).

Solution: We check first to see if **F** is conservative: $M_y = 1$ and $N_x = 1$. Since $M_y = N_x$, then **F** is conservative, and it is not necessary to parametrize the sequence of line segments. Instead, we find ϕ and evaluate it by using the Fundamental Theorem of Line Integrals. We need a potential function. Note that

$$\int y\,dx = xy \quad \text{and} \quad \int (x+2y)\,dy = xy + y^2.$$

Thus, $\phi(x,y) = xy + y^2$ is the (probable) potential function. We check by finding $\nabla\phi$: $\phi_x = y$ and $\phi_y = x + 2y$. These are *M* and *N*, respectively, so $\phi(x,y) = xy + y^2$ is a correct potential function. Therefore,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= [xy + y^2]_{(1,3)}^{(8,2)} \\ &= \big((8)(2) + (2)^2\big) - \big((1)(3) + (3)^2\big) \\ &= 20 - 12 \\ &= 8.\end{aligned}$$

All of the intermediate points were ignored. We only needed the starting and ending point of the path.

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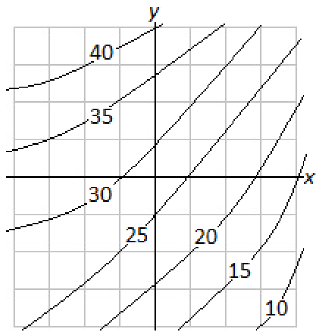
Example 48.4: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle 2x, 3y^2\rangle$ and *C* is given by $\mathbf{r}(t) = \langle t^2, 5t\rangle$ for $-1 \le t \le 3$.

Solution: Note that $M_y = 0$ and that $N_x = 0$. Since $M_y = N_x$, then **F** is conservative, and that $\phi(x,y) = x^2 + y^3$ is the potential function. Since **F** is conservative, the actual path of *C* is not relevant. We just need its two endpoints. When $t = -1$, we have $\mathbf{r}(-1) = \langle (-1)^2, 5(-1)\rangle = \langle 1, -5\rangle$, and when $t = 3$, we have $\mathbf{r}(3) = \langle (3)^2, 5(3)\rangle = \langle 9,15\rangle$. Note that $\langle 1, -5\rangle$ and $\langle 9,15\rangle$ are vectors, but if their feet are placed at the origin, then their heads point to the ordered pairs (1, –5) and (9,15). In this way, the point as ordered pairs can be inferred from a vector.

Therefore, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = [x^2 + y^3]_{(1,-5)}^{(9,15)} = ((9)^2 + (15)^3) - ((1)^2 + (-5)^3) = 3580.$$

Example 48.5: The contour map of $z = f(x,y)$ is below, for $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$. Suppose that vector field $\mathbf{F}(x,y) = \nabla f(x,y)$.



Evaluate the following:

a) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any path from (2,–1) to (–3,1).

b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any path from (–1,0) to (–2,3), then to (3,4)

c) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a circle of radius 2, centered at the origin.

Solution:

a) Since $\mathbf{F}(x,y) = \nabla f(x,y)$, then f is a potential function of the vector field $\mathbf{F}$, and $\mathbf{F}$ is conservative. Thus, $\mathbf{F}$ is path-independent, and only the starting and ending points of C are relevant. Note that from the contour map, we have $z = f(2,-1) = 20$ as the starting point, and $z = f(-3,1) = 35$ as the ending point. By the Fundamental Theorem of Line Integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = [f(x,y)]_{(2,-1)}^{(-3,1)}$$
$$= f(-3,1) - f(2,-1)$$
$$= 35 - 20$$
$$= 15.$$

b) Because $\mathbf{F}$ is conservative, only the starting and ending points of the path are relevant. Note that $f(-1,0) = 30$ and that $f(3,4) = 30$. Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 30 - 30 = 0$.

c) Since $\mathbf{F}$ is a conservative vector field and C is a closed simple loop, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.