25. Chain Rule

The Chain Rule is present in all differentiation. If $z = f(x, y)$ represents a two-variable function, then it is plausible to consider the cases when *x* and *y* may be functions of other variable(s). For example, consider the function $f(x, y) = x^2 + y^3$, where $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$. In such a case, we can find the derivative of f with respect to t by direct substitution, so that f is written as a function of t only, or we may use a form of the Chain Rule for multi-variable functions to find this derivative.

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Example 25.1: Given $f(x, y) = x^2 + y^3$, where $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$. Find $\frac{df}{dt}$.

Solution: Substitute $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$ into the function f:

$$f(x(t), y(t)) = (2t + 1)^2 + (3t^2 + 4t)^3$$

Now, f is a function of t only. Expand by multiplication:

$$f(t) = 4t^2 + 4t + 1 + 27t^6 + 108t^5 + 144t^4 + 64t^3$$

Thus, $f(t) = 27t^6 + 108t^5 + 144t^4 + 64t^3 + 4t^2 + 4t + 1$. Its derivative is found by applying the Power Rule to each term:

$$\frac{df}{dt} = f'(t) = 162t^5 + 540t^4 + 576t^3 + 192t^2 + 8t + 4.$$

Now, let's try a different approach. Keeping the *x* and *y* variables present, write the derivative of f using the Chain Rule:

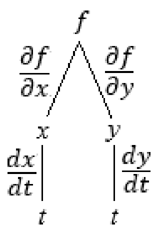
$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right) = (2x)(2) + (3y^2)(6t + 4).$$

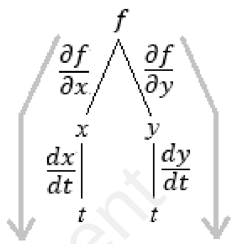
Now we substitute $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$ into the expression, and simplify:

$$\begin{aligned}\frac{df}{dt} &= (2x)(2) + (3y^2)(6t + 4) \\ &= \big(2(2t + 1)\big)(2) + (3(3t^2 + 4t)^2)(6t + 4) \\ &= 8t + 4 + 3(9t^4 + 24t^3 + 16t^2)(6t + 4) \\ &= 8t + 4 + 3(54t^5 + 180t^4 + 192t^3 + 64t^2) \\ &= 162t^5 + 540t^4 + 576t^3 + 192t^2 + 8t + 4.\end{aligned}$$

Both methods work, but the second method, by writing out all derivatives using all variables present, is more general, and also allows us to see patterns in how these derivatives are written.

A useful way to visualize the form of the Chain Rule is to sketch a derivative tree. In the previous example, we had f as a function of x and y, and then x and y as functions of t. Thus, we would write the tree as shown below. Then, the derivative form is found by multiplying along paths, and summing the separate paths:





$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

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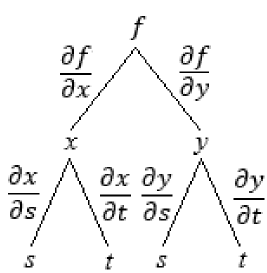
Example 25.2: Suppose $f(x, y) = 2xy^2$ and $x(s, t) = 3s - 2t$ and $y(s, t) = s^2 + 4t$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution: Note that f is a function of x and y, and that x and y are both functions of s and t. The derivative tree is shown below, with partial derivative notation attached to the "limbs":



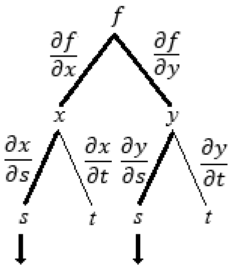
For example, the form of the partial derivative of f with respect to s is

$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right).$$

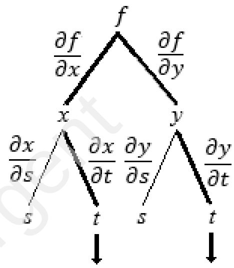
In a similar way, the form of the partial derivative of f with respect to t is

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right).$$

The tree helps us visualize the form of the derivatives:



$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$



$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

Thus, to find $\frac{\partial f}{\partial s}$, we have

$$\begin{aligned} \frac{\partial f}{\partial s} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right) \\ &= (2y^2)(3) + (4xy)(2s) \\ &= 6y^2 + 8xys \\ &= 6(s^2+4t)^2 + 8(3s-2t)(s^2+4t)s \qquad \begin{cases} y = s^2 + 4t \\ x = 3s - 2t \end{cases} \end{aligned}$$

This is simplified to

$$\frac{\partial f}{\partial s} = 30s^4 - 16s^3t + 144s^2t - 64st^2 + 96t^2.$$

To find $\frac{\partial f}{\partial t}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) \\
&= (2y^2)(-2) + (4xy)(4) \\
&= -4y^2 + 16xy \\
&= -4(s^2+4t)^2 + 16(3s-2t)(s^2+4t) \qquad \begin{cases} y = s^2 + 4t \\ x = 3s - 2t \end{cases}
\end{aligned}$$

This simplifies to

$$\frac{\partial f}{\partial t} = -4s^4 + 48s^3 - 64s^2t + 192st - 192t^2.$$

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Example 25.3: Let $y = g(u, v, w)$ and let *u*, *v* and *w* be functions of *m* and *n*. Suppose that $\frac{\partial g}{\partial m} = 19$, $\frac{\partial g}{\partial u} = 4$, $\frac{\partial g}{\partial v} = 2$, $\frac{\partial g}{\partial w} = 3$, $\frac{\partial u}{\partial m} = 5$, $\frac{\partial u}{\partial n} = 11$, $\frac{\partial v}{\partial n} = 1$, $\frac{\partial w}{\partial m} = -5$ and $\frac{\partial w}{\partial n} = 12$. Find $\frac{\partial v}{\partial m}$.

Solution: Using a derivative tree (or recognizing the pattern of the Chain Rule), we have

$$\frac{\partial g}{\partial m} = \left(\frac{\partial g}{\partial u}\right)\left(\frac{\partial u}{\partial m}\right) + \left(\frac{\partial g}{\partial v}\right)\left(\frac{\partial v}{\partial m}\right) + \left(\frac{\partial g}{\partial w}\right)\left(\frac{\partial w}{\partial m}\right).$$

By substitution, we have

$$\begin{aligned}
19 &= (4)(5) + (2)\left(\frac{\partial v}{\partial m}\right) + (3)(-5) \\
19 &= 20 + 2\left(\frac{\partial v}{\partial m}\right) - 15 \\
\frac{19 - 20 + 15}{2} &= \frac{\partial v}{\partial m} \\
\frac{\partial v}{\partial m} &= 7.
\end{aligned}$$

Note that we did not need to use the information provided for $\frac{\partial u}{\partial n}$, $\frac{\partial v}{\partial n}$ and $\frac{\partial w}{\partial n}$.

Implicit Differentiation can be performed by employing the chain rule of a multivariable function. Often, this technique is much faster than the "traditional" direct method seen in single-variable calculus can be applied to functions of many variables with ease.

Example 25.4: Use implicit differentiation to find $\frac{dy}{dx}$ where $x^2y + y^3 = x^4$.

Solution: The "traditional" method is to differentiate each term in place, with respect to x. Note that the product rule is used on the first term, where $\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy$:

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(x^4),$$

$$\text{which gives} \qquad x^2\frac{dy}{dx} + 2xy + 3y^2\frac{dy}{dx} = 4x^3.$$

Then algebraically isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + 3y^2) = 4x^3 - 2xy, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{4x^3 - 2xy}{x^2 + 3y^2}.$$

To use the Chain Rule, rewrite the equation $x^2y + y^3 = x^4$ with all terms to one side:

$$x^2y + y^3 - x^4 = 0.$$

Call the left side $F(x, y) = x^2y + y^3 - x^4$. We seek $\frac{dy}{dx}$, so differentiate both sides with respect to x. Using the Chain Rule, the derivative of F with respect to x is written $\left(\frac{\partial F}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right)$. Note that the right side gives us $\frac{d}{dx}0 = 0$. We have

$$\left(\frac{\partial F}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right) = 0.$$

Now, $\frac{\partial F}{\partial x} = 2xy - 4x^3$ and $\frac{\partial F}{\partial y} = x^2 + 3y^2$. Furthermore, $\frac{dx}{dx} = 1$, and $\frac{dy}{dx}$ is the unknown. Make the substitutions and solve for the unknown:

$$
\begin{aligned}
(2xy - 4x^3)(1) + (x^2 + 3y^2)\frac{dy}{dx} &= 0 \\
(x^2 + 3y^2)\frac{dy}{dx} &= 4x^3 - 2xy \\
\frac{dy}{dx} &= \frac{4x^3 - 2xy}{x^2 + 3y^2}.
\end{aligned}
$$

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In general, if x and y are implicitly related, collect all terms to one side and call the collected expression $F(x, y)$. Thus,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \qquad \text{and} \qquad \frac{dx}{dy} = -\frac{F_y}{F_x}.$$

This is true for implicit functions of three or more variable, too.

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Example 25.5: Given $xy^2z + 3xz^3 = yz^6$, find $\frac{dy}{dz}$.

Solution: Call $F(x, y, z) = xy^2z + 3xz^3 - yz^6$. Using the formula $\frac{dy}{dz} = -\frac{F_z}{F_y}$, we have

$$F_z = xy^2 + 9xz^2 - 6yz^5 \qquad \text{and} \qquad F_y = 2xyz - z^6.$$

Thus,

$$\frac{dy}{dz} = -\frac{F_z}{F_y} = -\left(\frac{xy^2 + 9xz^2 - 6yz^5}{2xyz - z^6}\right) = \frac{6yz^5 - xy^2 - 9xz^2}{2xyz - z^6}.$$

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