Notes, Tuesday 1-13-26

Limits

how does a function behave <u>near</u> a point?

ex: $f(x) = x^2$

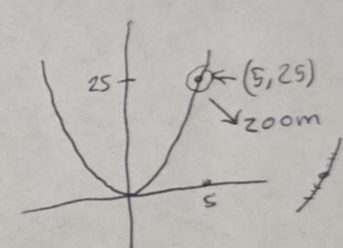
$f(5) = 25 \rightarrow$ AT $x=5$, $f(x)=25$

What about near $x=5$?

x	$f(x)$
4.9	24.01
4.99	24.9001
4.999	24.990001
$\downarrow$	$\downarrow$
5	25

limit, as x approaches 5 from below (or left), is 25

$$\lim_{x \to 5^-} f(x) = 25$$



x	$f(x)$
5.1	26.01
5.01	25.1001
5.001	25.010001
$\downarrow$	$\downarrow$
5	25

limit, as x approaches 5 from above (or right), is 25

$$\lim_{x \to 5^+} f(x) = 25$$

$$g(x) = \frac{x^2-4}{x-2} \quad (x \neq 2) \qquad (x \wedge 2 - 4)/(x-2)$$

$\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

what is $g(2)$? no answer! g is not defined at $x=2$ (0 in denom.)

$g(x) = x+2$, $x \neq 2$

the restriction is still in place.

what is the behavior near $x=2$? (set up tables)

x	$g(x)$
1.9	3.9
1.99	3.99
1.999	3.999

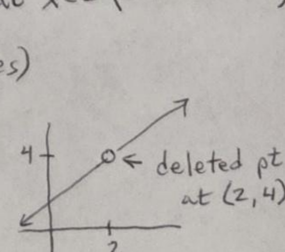
$$\lim_{x \to 2^-} g(x) = 4$$

"Left hand limit"
LH

x	$g(x)$
2.1	4.1
2.01	4.01
2.001	4.001

$$\lim_{x \to 2^+} g(x) = 4$$

"right hand limit"
RH



General limit:
If LH & RH limits agree, then the general limit is:

$$\lim_{x \to 2} g(x) = 4$$

$h(x) = \frac{1}{x} \qquad x \neq 0$

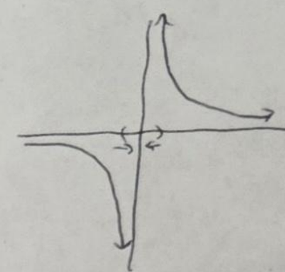
$\lim_{x \to 3^-} h(x) = \frac{1}{3}$

$\lim_{x \to 0^-} h(x) = -\infty$

x	$h(x)$
$-.1$	-10
$-.01$	-100
$-.001$	-1000
$\downarrow$	$\downarrow$
0^-	$-\infty$

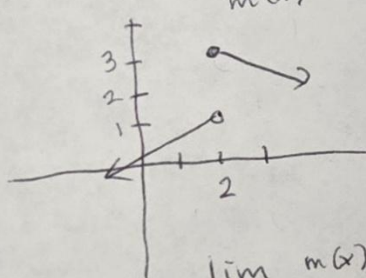
$\lim_{x \to 0^+} h(x) = +\infty$

x	$h(x)$
$.1$	10
$.01$	100
$.001$	1000
$\downarrow$	$\downarrow$
0^+	$+\infty$



the general limit D.N.E.

$m(x)$

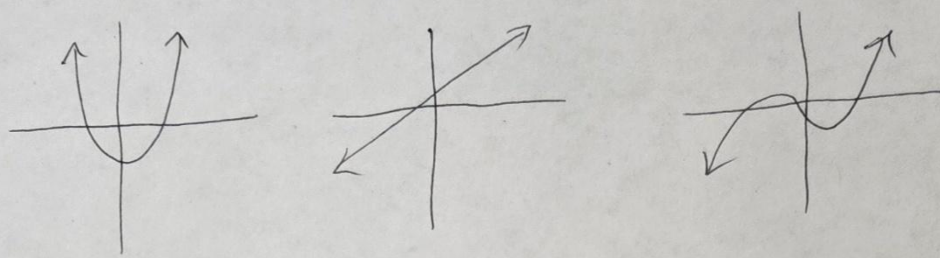


$\lim_{x \to 2^-} m(x) = 1 \qquad \lim_{x \to 2^+} m(x) = 3$

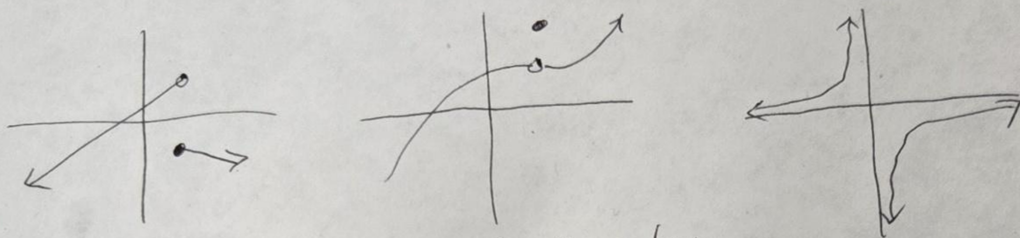
General: $\lim_{x \to 2} m(x) =$ D.N.E.

Thursday, 1-15-26

Limits and Continuity



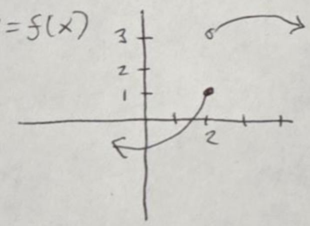
These are all continuous



These are not continuous

goal: describe, using limits, where a graph is continuous (or not), we can do this very specifically. "where and why".

$y=f(x)$



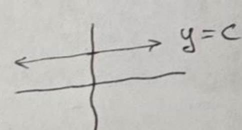
discontinuous at $x=2$, continuous elsewhere.

$$\left.\begin{aligned}\lim_{x\to 2^-} f(x) &= 1\\ \lim_{x\to 2^+} f(x) &= 3\end{aligned}\right\}$$ General limit: $\lim_{x\to 2} f(x) = \text{DNE}$

If LH lim = RH lim, then general limit exists.

If LH lim $\neq$ RH lim, then the general limit DNE

$\lim_{x\to a} c = c$



$$\lim_{x\to 4} x^2-3x+7 = \underbrace{\lim_{x\to 4} x^2}_{16} - \underbrace{3\lim_{x\to 4} x}_{-3(4)} + \underbrace{\lim_{x\to 4} 7}_{7} = 16-12+7 = 11$$

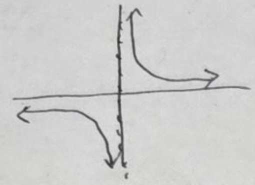
$$\lim_{x\to 2} \frac{x^2+5}{2x+7} = \frac{9}{11}$$

For all polynomials, exponentials, logarithms, sine/cosine functions, just plug in the "a" value

these are well-behaved functions with no "breaks".

Why you can't divide by zero:

$f(x) = \frac{1}{x}$



we call this "singularity"

x	$\frac{1}{x}$
2	1/2
5	1/5
10	1/10
100	1/100
1000	1/1000
↓	↓
∞	0

$\lim_{x \to \infty} \frac{1}{x} = 0$

x	$\frac{1}{x}$
1	1
1/2	2
1/5	5
1/10	10
1/100	100
1/1000	1000
↓	↓
0	∞

$\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} \to \frac{0}{0}$ "indeterminate."

we ~~can~~ look at LH, RH limits.

$\lim_{x \to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}} \to -6$

Y = (X^2 + 4X - 5)/(2X^2 + X + 1)

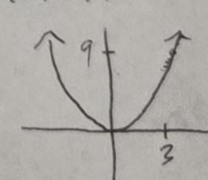
$\lim_{x \to \infty} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$ horiz. asymptote

Continuity:

$\lim_{x \to a} f(x) = f(a)$

If ① limit exists as $x \to a$, and
② $f(a)$ exists, and
③ they are equal,
then $f(x)$ is continuous at $x = a$.

$f(x) = x^2$



Q: is this continuous at $x=3$?

① $\lim_{x \to 3} f(x) = 9$

② $f(3) = 9$

③ $9 = 9$ ☺

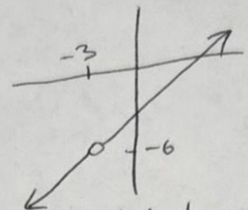
continuous at $x=3$

$$f(x) = \frac{x^2 - 9}{x + 3}$$ continuous at $x = -3$, if not, why?

① $\lim_{x \to -3} f(x) = -6$

② $f(-3)$ DNE

③ $\lim_{x \to -3} f(x) \neq f(-3)$



deleted point discontinuity.

$$g(x) = \begin{cases} \frac{1}{2}x + 3 & \text{for } x < -2 \\ x - 1 & \text{for } x \geq -2 \end{cases}$$

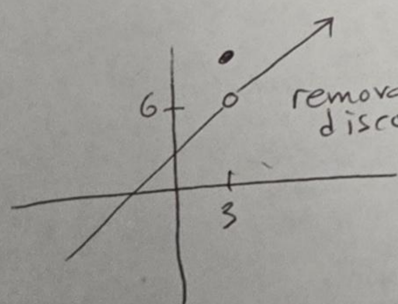
① $\lim_{x \to -2^-} g(x) = 2$, $\lim_{x \to -2^+} g(x) = -3 \Rightarrow \lim_{x \to -2} g(x)$ DNE

② $g(-2) = -3$

③

it is discontinuous at $x = -2$ because $\lim_{x \to -2}$ DNE

"jump"



removable discontinuity.