MAT 267 – Exam 2 – Surgent – October 19, 2023 KEY

1. The derivate of $f(x,y,z) = 2xy - 3yz$ in the direction of $\mathbf{v} = \langle 0,3,-4\rangle$ at the point (2,4,0) is

$$\nabla f = \langle 2y, 2x - 3z, -3y\rangle, \rightarrow \nabla f(2,4,0) = \langle 8,4,-12\rangle$$

Direction as a unit vector is $v = \langle 0, \frac{3}{5}, -\frac{4}{5}\rangle$. Thus, slope is $\langle 8,4,-12\rangle \cdot \langle 0,\frac{3}{5},-\frac{4}{5}\rangle = \frac{12}{5} - \left(-\frac{48}{5}\right) = \frac{60}{5} = 12$

Ans(1) C

A. 60 **B.** 5 **C.** 12 **D.** -60

2. The equation of the tangent plane of $f(x,y) = 3x^2 - \frac{1}{y}$ at $x_0 = 2$ and $y_0 = -1$ is

Note that at $f(2,-1) = 3(2)^2 - \frac{1}{-1} = 12 + 1 = 13.$

$$\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 6x, \frac{1}{y^2}, -1\rangle \rightarrow \text{at } x = 2, y = -1 = \langle 12,1,-1\rangle$$

We have $12(x-2) + \big(y - (-1)\big) - (z - 13) = 0$, which is $12 + y - z - 10 = 0$ or $z = 12x + y - 10$

Ans(2) D

A. $z = 12x + y - 23$ **B.** $z = 12x + y + 13$
C. $z = 12x + y - 12$ **D.** $z = 12x + y - 10$

3. Evaluate $\int_0^2 \int_x^2 xy^3 \, dy \, dx$.

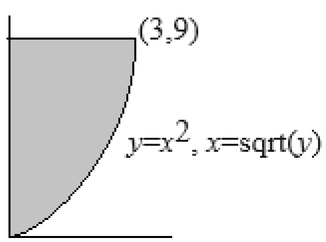
Inner integral: $\int_x^2 xy^3 \, dy = x\left(\frac{1}{4}y^4\right)_x^2 = x\left(\frac{1}{4}2^4 - \frac{1}{4}x^4\right) = x\left(4 - \frac{1}{4}x^4\right) = 4x - \frac{1}{4}x^5$

Outer integral: $\int_0^2 \left(4x - \frac{1}{4}x^5\right) dx = \left(2x^2 - \frac{1}{24}x^6\right)_0^2 = 2(2)^2 - \frac{1}{24}(2)^6 = 8 - \frac{64}{24} = \frac{16}{3}$

Ans(3) D

A. $5/3$ **B.** $-8/3$ **C.** $8/3$ **D.** $16/3$

4. Reverse the order of integration of $\int_0^3 \int_{x^2}^9 f(x,y) \, dy \, dx$.



Ans(4) C

A. $\int_{x^2}^9 \int_0^3 f(x,y) \, dx \, dy$ **B.** $\int_0^9 \int_0^{x^2} f(x,y) \, dx \, dy$
C. $\int_0^9 \int_0^{\sqrt{y}} f(x,y) \, dx \, dy$ **D.** $\int_0^3 \int_{\sqrt{y}}^9 f(x,y) \, dx \, dy$

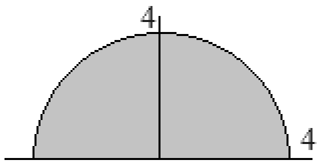
5. The steepest slope on $f(x,y) = xy^2 + x^2y$ at $x_0 = 3$ and $y_0 = 2$ is

$$\nabla f = \langle y^2 + 2xy, 2xy + x^2\rangle \rightarrow \nabla f(3,2) = \langle 16,21\rangle \text{ so slope} = |\nabla f| = |\langle 16,21\rangle| = \sqrt{16^2 + 21^2} = \sqrt{697}$$

Ans(5) D

A. $\langle 16,21\rangle$ **B.** $\langle y^2 + 2xy, 2xy + x^2\rangle$ **C.** 30 **D.** $\sqrt{697}$

6. The Cartesian bounds $0 \le y \le \sqrt{16-x^2}$ and $-4 \le x \le 4$ is equivalent to the polar bounds



Ans(6) A

A. $0 \le r \le 4, 0 \le \theta \le \pi$ **B.** $0 \le r \le 16, 0 \le \theta \le \pi$
C. $0 \le r \le 2, 0 \le \theta \le \pi$ **D.** $0 \le r \le 4, 0 \le \theta \le 2\pi$

7. The critical point of $g(x, y) = xy$ subject to the constraint $2x + y = 6$ is

Solve for y: $y = 6 - 2x$. Substitute: $g(x) = x(6-2x) = 6x - 2x^2$. Differentiate: $g' = 6 - 4x$. Critical value is $x = \frac{3}{2}$, so y=3 and $g\left(\frac{3}{2}, 3\right) = \frac{9}{2}$. Note that $6x - 2x^2$ is a parabola opening down, so this point is a max.

Ans(7) B

A. $\left(\frac{3}{2}, 3, \frac{9}{2}\right)$, min **B.** $\left(\frac{3}{2}, 3, \frac{9}{2}\right)$, max
C. $\left(-\frac{3}{2}, 3, -\frac{9}{2}\right)$, min **D.** $\left(-\frac{3}{2}, 3, -\frac{9}{2}\right)$, max

8. The domain of the function $h(x, y) = \sqrt{100 - x^2 - y^2}$ is

Set the inside ≥ 0. It is a filled-in circle of radius 10.

Ans(8) D

A. $0 \le x \le 10, 0 \le y \le 10$ **B.** $x + y \le 10$
C. $x^2 + y^2 \le 10$ **D.** $x^2 + y^2 \le 100$

9. Given $f(x, y) = x^3 + y^2 - 6xy$. Find all critical points and use the second derivative "D test" to classify them as minimum, maximum or saddle points. Give the point(s) with all coordinates.

$$\begin{matrix} f_x = 3x^2 - 6y \\ f_y = 2y - 6x \end{matrix} \rightarrow \begin{matrix} 3x^2 - 6y = 0 \\ 2y - 6x = 0 \end{matrix} \rightarrow y = 3x \rightarrow 3x^2 - 6(3x) = 0 \rightarrow 3x^2 - 18x = 0$$

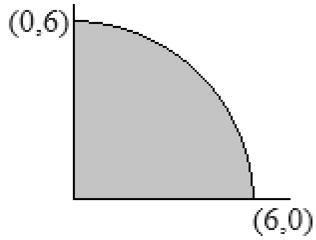
Factor: $3x^2 - 18x = 0 \rightarrow 3x(x-6) = 0 \rightarrow x = 0, x = 6$.

When $x = 0$, then $y = 0$ and $z = 0$, so (0,0,0) is a critical point.

When $x = 6$, then $y = 18$ and $z = -108$, so (6,18,-108) is a critical point.

$D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(2) - 6.$
At $x = 0$, $y = 0$, we have $D = -6$ so $D < 0$, so a saddle.
At $x = 6$, $y = 18$, we have $D = 36$ and $f_{xx} > 0$ so it is a min.

10. Evaluate $\int_0^6 \int_0^{\sqrt{36-x^2}} (1 + x^2 + y^2)\, dy\, dx$ exactly (no decimal equivalents).



This is a quarter circle region of radius 6, so $0 \leq r \leq 6, 0 \leq \theta \leq \frac{\pi}{2}$

Use Polar:

$$\int_0^{\pi/2} \int_0^6 (1 + r^2) r\, dr\, d\theta$$

Inner integral:

$$\int_0^6 (1 + r^2) r\, dr = \int_0^6 r + r^3\, dr = \left[\frac{1}{2} r^2 + \frac{1}{4} r^4\right]_0^6 = \frac{1}{2}(6)^2 + \frac{1}{4}(6)^4 = \frac{36}{2} + \frac{1296}{4} = 18 + 324 = 342$$

Outer integral:

$$\int_0^{\pi/2} 342\, d\theta = 342\left(\frac{\pi}{2}\right) = 171\pi.$$

11. A crystal in the shape of a pyramid with a square base is growing through accretion, adding 1 mm^3 per day. Find the rate at which the height of the crystal is changing at the moment the base is 4 mm, the height is 3 mm, and both base lengths are increasing by 0.03 mm/day. The volume of a square-based pyramid of base b and with height h is $V(b, h) = \frac{1}{3} b^2 h$.

$$\Delta V = V_b \Delta b + V_h \Delta h$$

$$\Delta V = \frac{2}{3} bh\, \Delta \mathrm{b} + \frac{1}{3} b^2\, \Delta h$$

$$1 = \frac{2}{3}(4)(3)(0.03) + \frac{1}{3}(4)^2 \Delta h$$

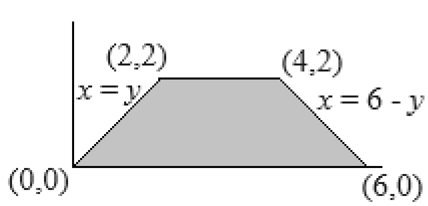
$$1 = 0.24 + \frac{16}{3} \Delta h$$

$$0.76 = \frac{16}{3} \Delta h$$

$$\frac{3}{16}(0.76) = \Delta h$$

About 0.1425 mm/day

12. Set up $\iint_R f(x, y)\, dA$, where R is the trapezoid in the xy-plane enclosed by the points (0,0), (6,0), (4,2) and (2,2). Do not solve, just set it up. For full credit, draw the region with all bounds labeled.



$$\int_0^2 \int_y^{6-y} f(x, y)\, dx\, dy$$