MAT 267 – Exam 1 – Surgent – September 14, 2023 KEY

Show all work and be neat. You may use the backside for scratch, but work written on the backside won't be looked at. Place all work you want graded on the front. You agree to abide by all aspects of the honor code while taking this exam and will not discuss the contents of this exam during the examination period.

PART I: Multiple Choice. Write the letter of your choice in the space provided at right. (6 pts each)

1. The equation $(x-4)^2 + y^2 + (z+1)^2 = 16$ represents a sphere. State its center and radius.

Ans(1) C

A. $(-4,0,1), 16$ **B.** $(4,0,-1), 16$ **C.** $(4,0,-1), 4$ **D.** $(-4,0,1), 4$

2. Given $\mathbf{u} = \langle 3,-1,4 \rangle$. The vector parallel to $\mathbf{u}$ and of length 5 is

Ans(2) A

Magnitude $|u| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$. Thus, unit vector is $\langle \frac{3}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}} \rangle$. Then multiply by 5.

A. $\langle \frac{15}{\sqrt{26}}, -\frac{5}{\sqrt{26}}, \frac{20}{\sqrt{26}} \rangle$ **B.** $\langle \frac{15}{\sqrt{24}}, -\frac{5}{\sqrt{24}}, \frac{20}{\sqrt{24}} \rangle$ **C.** $\langle 15,-5,20 \rangle$ **D.** $\langle -15,5,-20 \rangle$

3. At what angle do the planes $4x + 7y - z = 12$ and $5x + 3y - 2z = 8$ intersect?

Ans(3) D

Normals are $n_1 = \langle 4,7,-1 \rangle$ & $n_2 = \langle 5,3,-2 \rangle$. Angle is $\cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right) = \cos^{-1}\left(\frac{43}{\sqrt{66}\sqrt{38}}\right) = 30.84°$.

A. 36.33° **B.** 32.05° **C.** 38.85° **D.** 30.84°

4. Let $\mathbf{u} = \langle 6,k,1 \rangle$ and $\mathbf{v} = \langle 2,5,k \rangle$. For what values k are $\mathbf{u}$ and $\mathbf{v}$ obtuse?

Ans(4) A

Dot: $u \cdot v = 12 + 5k + k = 12 + 6k$. Want $12 + 6k < 0$ so $6k < -12$ and thus, $k < -2$.

A. $(-\infty,-2)$ **B.** $(-2,\infty)$ **C.** $(2,\infty)$ **D.** $(-\infty,2)$

5. Find the equation of the line passing through A = (4,2,–1) and B = (–3,1,5) such that $t = 0$ gives A and $t = 1$ gives B.

Ans(5) C

The direction vector is $AB = \langle -3-4, 1-2, 5-(-1) \rangle = \langle -7,-1,6 \rangle$.

A. $\langle 4-3t, 2+t, -1+5t \rangle, 0 \le t \le 1$ **B.** $\langle -3+4t, 1+2t, 5-t \rangle, 0 \le t \le 1$
C. $\langle 4-7t, 2-t, -1+6t \rangle, 0 \le t \le 1$ **D.** $\langle -3+7t, 1-t, 5-6t \rangle, 0 \le t \le 1$

6. Let $\mathbf{r}(t) = \langle 3t^2 + t, 2t-3, t^3 \rangle$ trace a curve in R^3. Find $\mathbf{T}(2)$.

Ans(6) C

$r'(t) = \langle 6t+1, 2, 3t^2 \rangle$ so $r'(2) = \langle 13,2,12 \rangle$. Also, $|r'(2)| = \sqrt{13^2 + 2^2 + 12^2} = \sqrt{307}$. So $T(2) = \frac{r'(2)}{|r'(2)|}$.

A. $\langle 13,2,12 \rangle$ **B.** $\langle 6,0,12 \rangle$ **C.** $\frac{1}{\sqrt{317}}\langle 13,2,12 \rangle$ **D.** $\frac{1}{\sqrt{180}}\langle 6,0,12 \rangle$

7. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel. What is true about $\mathbf{u} \times \mathbf{v}$?

Ans(7) A

A. It is $\mathbf{0}$ **B.** It is the product of their lengths
C. It points out of the page **D.** Not enough information

8. Where does the line $\langle 2-t, 1+3t, 3+4t\rangle$ intersect the plane $3x+5y-z=24$?

Ans(8) C

Substitute: $3(2-t)+5(1+3t)-(3+4t)=24$. Solve for t gives $t=2$. The evaluate to get the point.

A. $t=2$ **B.** $t=-2$ **C.** $(0,7,11)$ **D.** $(4,-5,-5)$

9. Given three points A = (0,4,3), B = (2,–1,1) and C = (1,2,2) in R^3.

a) Find the equation of the plane passing through these points. Any equivalent form of the correct answer will be accepted.

Form two vectors: $AB=\langle 2,-5,-2\rangle$ and $AC=\langle 1,-2,-1\rangle$. Cross them to get the normal: $AB\times AC=\langle 1,0,1\rangle$. Assemble: $1(x-2)+0(y+5)+1(z-3)=0$. Simplified, $x+z=3$.

b) Find the area of the triangle formed by ABC.

Area of triangle = half the magnitude of the cross product: $\frac{1}{2}|\langle 1,0,1\rangle|=\frac{1}{2}\sqrt{2}$.

10. Let $\mathbf{r}(t)=\langle 4t, 3\cos t, 3\sin t\rangle$ trace a curve in R^3.

a) Find the length of the arc traced over $0\le t\le \pi$.

$r'(t)=\langle 4,-3\sin t, 3\cos t\rangle$. Thus, $|r'(t)|=\sqrt{4^2+(-3\sin t)^2+(3\cos t)^2}=\sqrt{16+9}=\sqrt{25}=5$. Arc length is $\int_0^\pi 5\,dt=5\pi$.

b) Find the equation (in vector form) of the tangent line to $\mathbf{r}$ at $t=\frac{\pi}{2}$.

Pass through vector/point is $r\left(\frac{\pi}{2}\right)=\langle 4\left(\frac{\pi}{2}\right), 3\cos\left(\frac{\pi}{2}\right), 3\sin\left(\frac{\pi}{2}\right)\rangle=\langle 2\pi,0,3\rangle$.

Direction vector is $r'(t)=\langle 4,-3\sin t, 3\cos t\rangle$ so $r'\left(\frac{\pi}{2}\right)=\langle 4,-3\sin\left(\frac{\pi}{2}\right), 3\cos\left(\frac{\pi}{2}\right)\rangle=\langle 4,-3,0\rangle$.

Line is $\langle 2\pi,0,3\rangle+t\langle 4,-3,0\rangle$.

11. Starting at the origin, an airplane is flying toward the point (10,9). The actual landing strip is at (12,5). The plane is allowed one right-angle turn in order to arrive at the landing strip. Find the point at which the plane makes this turn (assume this is a special plane that can do right-angle turns instantaneously in air).

Let $u=\langle 10,9\rangle$ and $v=\langle 12,5\rangle$. Vector v is the hypotenuse, vector u will form the adjacent leg. Project the hypotenuse onto the adjacent: v onto u:

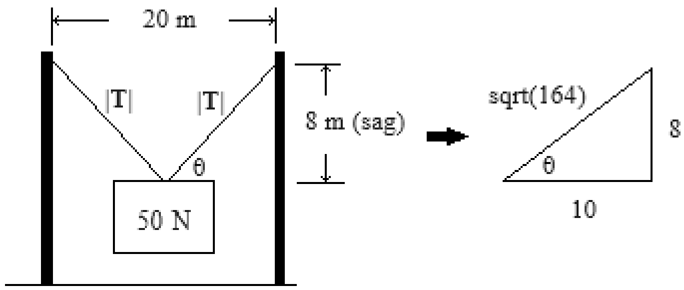
$$proj_u v=\left(\frac{u\cdot v}{u\cdot u}\right)u=\frac{165}{181}u=\left(\frac{1650}{181},\frac{1485}{181}\right)\approx(9.12, 8.20).$$

12. The position of an object along a path is given by $\mathbf{r}(t)=\langle t^2, 5t, t^3\rangle$. Find the object's speed at $t=2$.

Speed = $|r'(t)|$. We have $r'(t)=\langle 2t, 5, 3t^2\rangle$, so $r'(2)=\langle 4,5,12\rangle$. Thus, $|\langle 4,5,12\rangle|=\sqrt{4^2+5^2+12^2}=\sqrt{185}$ or about 13.6.

13. An object hangs from two cables symmetrically as shown below. Find the tension within each cable.

Place θ where shown and form a right triangle with hypotenuse:



We have $2|\mathbf{T}|\sin\theta = 50$ so that $|\mathbf{T}| = \frac{25}{\sin\theta}$. From the triangle, $\sin\theta = \frac{8}{\sqrt{164}}$, so $|\mathbf{T}| = \frac{25}{8/\sqrt{164}} \approx 40.02\ N$. You never actually need to determine the angle.